A Ten-Face Non-Edge-Sharing Wing Set on the Regular Icosahedron and a Decagonal Equatorial Balance

YoungJune Jeon
GeoWind

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Abstract

We formalize a ten-face triangular "wing" set on a regular icosahedron under a vertex labeling $\{N, S, U_1, \ldots, U_5, L_1, \ldots, L_5\}$ with rotation axis NS. The wing faces satisfy: (i) each face is an isosceles 36°–36°–108° triangle with a 36° angle anchored at a pole (N or S); (ii) distinct faces may share vertices but share no edges; and (iii) a natural equatorial cross-section yields a perfectly balanced regular decagon. We derive a closed form for the decagon radius, $R = \frac{\varphi}{2}\ell$, where ℓ is the icosahedron edge length and $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Beyond the geometric results, we interpret the "ten"-face closure as a symmetry-consistent design principle for a pole-anchored wing layout and provide a reproducible construction workflow.

1 Introduction

Regular polyhedra provide highly constrained geometric structures where design rules can be stated as axioms and verified as theorems. This paper defines and analyzes a specific triangular face set motivated by a GeoWind wing rule-set. Our first goal is mathematical: a precise definition and closed-form consequences for angles, symmetry, and a cross-sectional balance. Our second goal is design-facing: to express the same structure as a reproducible, checkable rule-set suitable for implementation in CAD/parametric geometry.

At a high level, the construction selects ten "golden" isosceles triangles (angles 36°–36°–108°) whose small angle is anchored at one of two poles on the rotation axis, and whose edges are pairwise disjoint (edge non-sharing). When all ten faces are present, their pole-opposite edges induce a regular decagon on the equatorial plane with a radius determined by the golden ratio.

2 Preliminaries

Let $\varphi = \frac{1+\sqrt{5}}{2}$ denote the golden ratio, satisfying $\varphi^2 = \varphi + 1$. Let I be a regular icosahedron with edge length ℓ and center O. Choose opposite vertices N and S and call the line NS the rotation axis.

2.1 A standard coordinate model

For proofs that require explicit distance computations, we use the standard coordinate model of the icosahedron with edge length 2 whose vertices are

$$(0, \pm 1, \pm\varphi), \quad (\pm 1, \pm\varphi, 0), \quad (\pm\varphi, 0, \pm 1). \tag{1}$$

Scaling by $\ell/2$ converts this model to edge length ℓ.

3 Labeling and Wing-Face Axioms

Definition 1 (Vertex labeling)**.** *Label the 12 vertices of the icosahedron as*

$$V = \{N, S, U_1, \ldots, U_5, L_1, \ldots, L_5\},$$

where indices are taken modulo 5*. The axis is the line NS.*

Figure 1 shows the labeled icosahedron with the rotation axis NS and the two vertex rings.

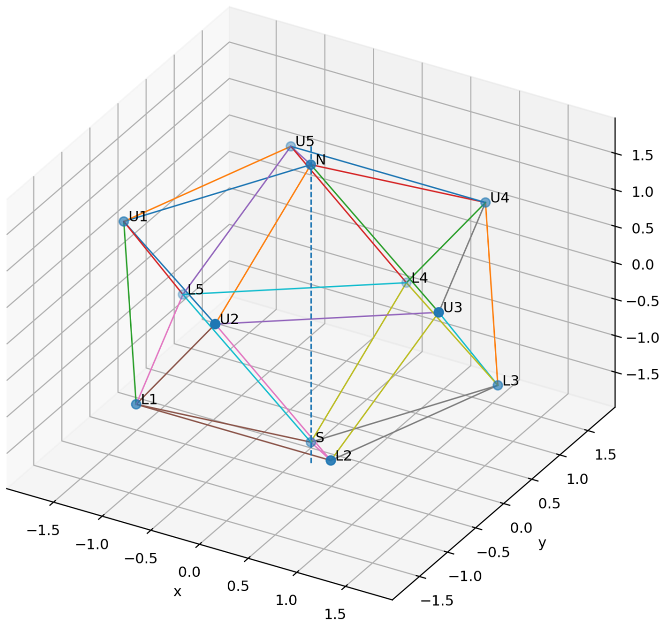

Figure 1: Labeled regular icosahedron used throughout the paper. The rotation axis is the line NS (aligned with the z-axis). The upper-ring vertices are labeled $U_1, \ldots, U_5$ and the lower-ring vertices $L_1, \ldots, L_5$.

Definition 2 (GeoWind ten-face wing set)**.** *Define the wing-face set $\mathcal{F}$ as the following ten triangles:*

$$F_S(i) = \triangle(S, U_i, L_i), \qquad i = 1, \ldots, 5,$$
$$F_N(i) = \triangle(N, U_i, L_{i-1}), \qquad i = 1, \ldots, 5,$$

with $L_0 := L_5$.

Axiom 1 (Edge non-sharing; vertex sharing allowed)**.** *For any two distinct faces $F_a \neq F_b \in \mathcal{F}$,*

$$E(F_a) \cap E(F_b) = \varnothing,$$

i.e., no edge is shared between two faces. Vertex sharing is allowed.

Axiom 2 (Non-intersection (geometric layout))**.** *The interiors of distinct faces do not intersect: for $F_a \neq F_b \in \mathcal{F}$,*

$$\mathrm{int}(F_a) \cap \mathrm{int}(F_b) = \varnothing.$$

The ten-face wing set $\mathcal{F}$ embedded in the labeled icosahedron is illustrated in Figure 2.

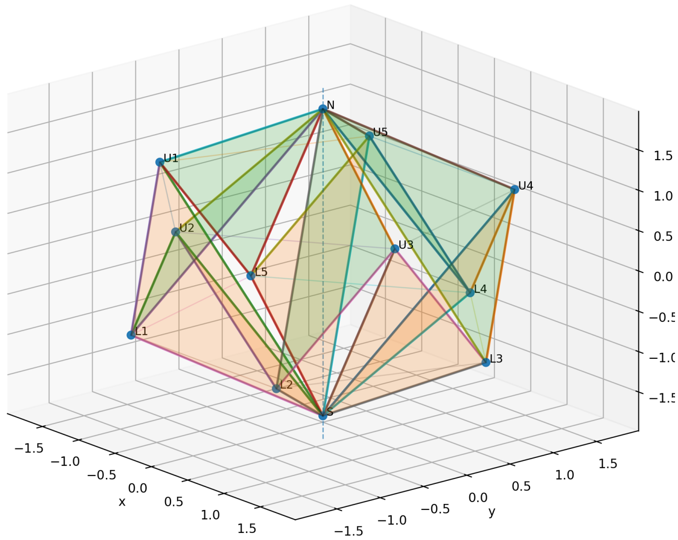

Figure 2: The GeoWind ten-face wing set $\mathcal{F}$ embedded in the labeled icosahedron. South-anchored faces are $F_S(i) = \triangle(S, U_i, L_i)$ and north-anchored faces are $F_N(i) = \triangle(N, U_i, L_{i-1})$ (indices modulo 5). Faces are shown translucent; vertex sharing is allowed, but no two faces share an edge.

4 Main Results

4.1 Angle structure via a golden-distance lemma

Lemma 1 (Edge and diagonal distances in the standard model)**.** *In the standard coordinate model* (1) *(edge length* 2*), (i) adjacent vertices have distance* 2*, and (ii) for an appropriate choice of opposite poles N, S and a corresponding upper vertex U, the distance between a pole and an adjacent upper vertex is* 2*, whereas the distance between the opposite pole and that upper vertex equals 2φ. Equivalently, after scaling to edge length ℓ, the corresponding distances are ℓ and $\varphi\ell$.*

Proof. For (i), one may verify adjacency by direct computation for a representative pair. Let $A = (0, 1, \varphi)$ and $B = (1, \varphi, 0)$; then

$$\|A - B\|^2 = (0 - 1)^2 + (1 - \varphi)^2 + (\varphi - 0)^2 = 1 + (1 - 2\varphi + \varphi^2) + \varphi^2.$$

Using $\varphi^2 = \varphi + 1$, the expression becomes

$$1 + (1 - 2\varphi + \varphi + 1) + (\varphi + 1) = 4,$$

hence $\|A - B\| = 2$. Similar checks generate the full edge set by symmetry.

For (ii), take poles $N = (0, 1, \varphi)$ and $S = (0, -1, -\varphi)$ (these are opposite in (1)). Let $U = (1, \varphi, 0)$, which is adjacent to N by (i). Then

$$\|N - U\| = 2, \qquad \|S - U\|^2 = (0 - 1)^2 + (-1 - \varphi)^2 + (-\varphi - 0)^2.$$

Compute

$$\|S-U\|^2 = 1 + (1 + 2\varphi + \varphi^2) + \varphi^2 = 2 + 2\varphi + 2\varphi^2.$$

Substituting $\varphi^2 = \varphi + 1$ gives

$$\|S-U\|^2 = 2 + 2\varphi + 2(\varphi + 1) = 4 + 4\varphi = 4\varphi^2,$$

so $\|S-U\| = 2\varphi$. Scaling the coordinate model by factor $\ell/2$ scales all distances by $\ell/2$, yielding ℓ and $\varphi\ell$. □

Lemma 2 (Cosine-law characterization of the golden gnomon)**.** *Let a triangle have side lengths* $(\ell, \ell, \varphi\ell)$*. Then its angles are* $(36°, 36°, 108°)$*.*

Proof. Let the equal sides have length ℓ, and let the base have length $\varphi\ell$. By the law of cosines, the angle θ opposite the base satisfies

$$\cos\theta = \frac{\ell^2 + \ell^2 - (\varphi\ell)^2}{2\ell \cdot \ell} = \frac{2 - \varphi^2}{2}.$$

Using $\varphi^2 = \varphi + 1$ gives

$$\cos\theta = \frac{2 - (\varphi + 1)}{2} = \frac{1 - \varphi}{2} = -\cos 72°,$$

hence $\theta = 108°$. The remaining two angles are equal and sum to $72°$, so each is $36°$. □

Theorem 1 (Golden-triangle shape: $36°–36°–108°$)**.** *Every face $F \in \mathcal{F}$ is an isosceles triangle with interior angles* $(36°, 36°, 108°)$*. Moreover, for $F_S(i)$ a $36°$ angle occurs at the pole S, and for $F_N(i)$ a $36°$ angle occurs at N.*

Proof. Fix i and consider $F_S(i) = \triangle(S, U_i, L_i)$. In the regular icosahedron, S is adjacent to L_i and U_i is adjacent to L_i, so

$$|SL_i| = \ell, \qquad |U_iL_i| = \ell.$$

By Lemma 1, the distance from S to the corresponding upper vertex is $\varphi\ell$, so

$$|SU_i| = \varphi\ell.$$

Thus $F_S(i)$ has side lengths $(\ell, \ell, \varphi\ell)$, and by Lemma 2 its angles are $(36°, 36°, 108°)$ with a $36°$ angle at S.

For $F_N(i) = \triangle(N, U_i, L_{i-1})$, N is adjacent to U_i and U_i is adjacent to L_{i-1}, hence

$$|NU_i| = \ell, \qquad |U_iL_{i-1}| = \ell.$$

Again by Lemma 1, the remaining side has length $\varphi\ell$, so the same angle conclusion follows, with a $36°$ angle anchored at N. □

4.2 Edge non-sharing and maximality

Theorem 2 (No shared edges within the ten faces)**.** *The ten faces in $\mathcal{F}$ satisfy Axiom 1: no two distinct faces share an edge.*

Proof. Each south face $F_S(i)$ has edge set $\{SU_i,\ SL_i,\ U_iL_i\}$. For $i \neq j$ these edges have different endpoints and thus cannot coincide. Each north face $F_N(i)$ has edge set $\{NU_i,\ NL_{i-1},\ U_iL_{i-1}\}$, and the same endpoint argument applies.

It remains to compare a south edge with a north edge. Any south edge incident to S cannot equal a north edge incident to N. A south edge of type U_iL_i cannot equal a north edge of type U_jL_{j-1} because equality would require $U_i = U_j$ and $L_i = L_{j-1}$, hence $i = j = j - 1 \pmod 5$, impossible. Therefore no edges are shared. □

Theorem 3 (Maximality under pole-anchored edge non-sharing). *Consider pole-anchored wing triangles of the form $\triangle(S, U_i, \cdot)$ whose edges must be pairwise non-shared. Then at most five such triangles can be anchored at S. Likewise, at most five can be anchored at N. Consequently, under pole anchoring at both poles and edge non-sharing, a construction can contain at most ten faces.*

Proof. Any triangle anchored at S using an upper vertex U_i necessarily contains the edge SU_i. Under edge non-sharing, no two such triangles may use the same edge SU_i. Since there are only five distinct upper vertices $U_1, \ldots, U_5$, there are only five distinct edges SU_i, hence at most five edge-disjoint pole-anchored triangles can be anchored at S. The same argument holds for N. □

4.3 Equatorial decagon and closed-form radius

Definition 3 (Equatorial plane and cross-edge midpoints). *Let the equatorial plane be*

$$\Pi := \{x \in \mathbb{R}^3 : (x - O) \cdot (N - S) = 0\},$$

i.e., the plane through O perpendicular to axis NS. For a face F, let $c(F)$ denote the edge not incident to the corresponding pole (the "cross-edge"). Define the representative point

$$p(F) := \mathrm{mid}(c(F)),$$

the midpoint of the cross-edge.

Proposition 1 (Cross-edges for the ten faces). *For the faces in Definition 2,*

$$c(F_S(i)) = U_i L_i, \qquad c(F_N(i)) = U_i L_{i-1}.$$

Proof. In $F_S(i) = \triangle(S, U_i, L_i)$ the two edges incident to pole S are SU_i and SL_i, hence the remaining edge is $U_i L_i$. The north case is identical. □

Theorem 4 (Perfect circular balance: regular decagon on Π). *The set $\{p(F) \mid F \in \mathcal{F}\} \subset \Pi$ consists of ten points lying on a single circle centered at O, forming a regular decagon with 36° angular spacing.*

Proof. By Proposition 1, each $p(F)$ is the midpoint of a U–L edge. By symmetry about the axis and the central plane through O orthogonal to NS, each such midpoint lies in Π, so $p(F) \in \Pi$.

Next, consider the two midpoint sets

$$P_1 = \{\mathrm{mid}(U_i L_i)\}_{i=1}^{5}, \qquad P_2 = \{\mathrm{mid}(U_i L_{i-1})\}_{i=1}^{5}.$$

Rotation by 72° about axis NS maps $U_i \mapsto U_{i+1}$ and $L_i \mapsto L_{i+1}$, so it cyclically permutes P_1 and also P_2. Hence each of P_1 and P_2 forms a regular pentagon on a circle centered at O in Π.

To show that $P_1 \cup P_2$ is a regular decagon, it suffices to show a 36° phase offset between one point in P_1 and a neighboring point in P_2. This is established in Proposition 2. Therefore the two pentagons interlace into a regular decagon with 36° spacing. □

Proposition 2 (Phase offset: $\cos 36^\circ = \varphi/2$). *In the standard model* (1) *(edge length* 2*), let*

$$M = \mathrm{mid}\big((0, 1, \varphi), (1, \varphi, 0)\big), \qquad M' = \mathrm{mid}\big((0, 1, \varphi), (-1, \varphi, 0)\big).$$

Then $\angle MOM' = 36^\circ$ and

$$\frac{M \cdot M'}{\|M\| \, \|M'\|} = \frac{\varphi}{2}.$$

Proof. Compute

$$M = \left(\frac{1}{2}, \frac{1+\varphi}{2}, \frac{\varphi}{2}\right), \qquad M' = \left(-\frac{1}{2}, \frac{1+\varphi}{2}, \frac{\varphi}{2}\right).$$

Their dot product is

$$M \cdot M' = -\frac{1}{4} + \left(\frac{1+\varphi}{2}\right)^2 + \left(\frac{\varphi}{2}\right)^2.$$

Using $\varphi^2 = \varphi + 1$,

$$\left(\frac{1+\varphi}{2}\right)^2 = \frac{1+2\varphi+\varphi^2}{4} = \frac{2+3\varphi}{4}, \qquad \left(\frac{\varphi}{2}\right)^2 = \frac{\varphi^2}{4} = \frac{\varphi+1}{4}.$$

Therefore

$$M \cdot M' = -\frac{1}{4} + \frac{2+3\varphi}{4} + \frac{\varphi+1}{4} = \frac{2+4\varphi}{4} = \frac{1+2\varphi}{2}.$$

Meanwhile, $\|M\| = \|M'\| = \varphi$ (see Theorem 5), so $\|M\|\,\|M'\| = \varphi^2$. Thus

$$\frac{M \cdot M'}{\|M\|\,\|M'\|} = \frac{(1+2\varphi)/2}{\varphi^2} = \frac{\varphi}{2},$$

which equals $\cos 36°$. Hence $\angle MOM' = 36°$. □

Theorem 5 (Closed-form radius $R = \frac{\varphi}{2}\ell$)**.** *Let R be the radius of the circle in Theorem 4. Then*

$$\boxed{R = \frac{\varphi}{2}\,\ell}.$$

Proof. In the standard model (1) (edge length 2), consider adjacent vertices $A = (0, 1, \varphi)$ and $B = (1, \varphi, 0)$. Their midpoint is

$$M = \left(\frac{1}{2},\ \frac{1+\varphi}{2},\ \frac{\varphi}{2}\right).$$

Using $\varphi^2 = \varphi + 1$,

$$\|M\|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1+\varphi}{2}\right)^2 + \left(\frac{\varphi}{2}\right)^2 = \varphi^2,$$

so $\|M\| = \varphi$. Hence the midpoint circle in the edge-length-2 model has radius φ. Scaling to edge length ℓ multiplies distances by $\ell/2$, giving $R = \varphi(\ell/2) = \frac{\varphi}{2}\ell$. □

Corollary 1. *If $\ell = 1\,\mathrm{m}$, then*

$$R = \frac{\varphi}{2}\ \mathrm{m} \approx 0.809016994\ \mathrm{m}.$$

Figure 3 illustrates the equatorial plane Π and the regular decagon formed by the representative points $p(F)$.

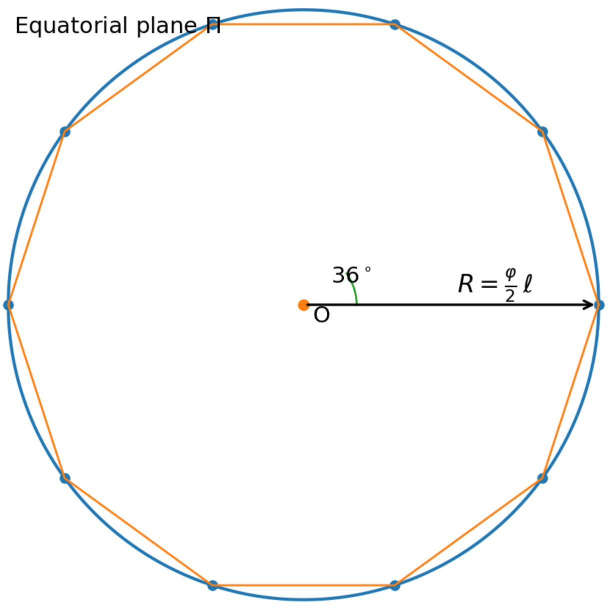

Figure 3: Equatorial plane Π (perpendicular to axis NS through center O). The ten representative points $p(F)$ (midpoints of the pole-opposite edges) lie on a circle centered at O and form a regular decagon of radius $R = \frac{\varphi}{2}\ell$.

5 Design Interpretation: Why "Ten"?

The number ten is not merely a count; it encodes a closure condition consistent with the icosahedron's fivefold rotational symmetry about the pole axis. On each pole, there are exactly five distinct edges SU_i (and NU_i). Under edge non-sharing, these edges act as five "ports" for pole-anchored faces (Theorem 3). The chosen index shift between south and north faces prevents edge duplication while interlacing the two pentagonal midpoint sets into a decagon (Theorem 4).

From a design standpoint, the equatorial decagon provides a compact diagnostic: if a CAD model implements the ten faces correctly, the midpoints of the pole-opposite edges must lie on a single circle and fall into a regular decagon with radius $\frac{\varphi}{2}\ell$. This offers a simple geometric "sanity check" independent of aerodynamic modeling.

6 Reproducible Construction Workflow (CAD/Parametric)

We summarize a practical procedure to generate and validate $\mathcal{F}$.

Algorithm 1 (Wing-face generation and validation)

1. **Create a labeled icosahedron.** Fix edge length ℓ and label vertices as in Definition 1.
2. **Generate faces.** Create $F_S(i) = \triangle(S, U_i, L_i)$ and $F_N(i) = \triangle(N, U_i, L_{i-1})$ for $i = 1, \dots, 5$.
3. **Edge non-sharing check.** List all edges of the ten triangles (unordered vertex pairs) and verify no duplicates.
4. **Angle check.** Verify side lengths $(\ell, \ell, \varphi\ell)$ and angles $(36^\circ, 36^\circ, 108^\circ)$ for each face.

5. **Equatorial decagon check.** Compute $p(F)$ for all faces. Confirm decagon geometry and radius $R = \frac{\varphi}{2}\ell$.
6. **Non-intersection check (optional).** Perform triangle–triangle intersection tests to numerically verify Axiom 2.

7 Related Geometric Context

The appearance of φ in the icosahedron is classical: coordinate realizations, vertex/edge relations, and duality with the dodecahedron all involve the golden ratio. The present contribution is a specific edge-disjoint, pole-anchored ten-face rule-set whose equatorial midpoints close into a decagon with a simple closed-form radius. For general background, see Coxeter and Grünbaum.

8 Conclusion

We defined a pole-anchored ten-face wing set on a regular icosahedron and proved: (i) each face is a $36° - 36° - 108°$ isosceles (golden) triangle; (ii) faces share no edges and at most ten such pole-anchored edge-disjoint faces can exist across both poles; and (iii) the pole-opposite edge midpoints form a regular decagon on the equatorial plane with radius $R = \frac{\varphi}{2}\ell$. These results provide a mathematical foundation and a practical validation workflow for geometry-driven design implementations.

References

[1] H. S. M. Coxeter, *Regular Polytopes*, Dover Publications.

[2] B. Grünbaum, *Convex Polytopes*, Springer.